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On Reformulating Quantum Mechanics and Stochastic Theory

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A stochastic theory approach is used to formulate the theory of quantum mechanical motion. Apart from giving a unifying point of view to quantum mechanics and stochastic theory, the new formulation is not limited to forces derivable from a potential. A nonlinear dynamical law is deduced in contradistinction to previous works in which *ad hoc* linear laws are postulated.

KEY WORDS: Nonlinear quantum and classical mechanics; stochastic motion.

1. INTRODUCTION

Many approaches have been attempted in order to formalize and extend the statistical concepts in quantum theory. One approach is von Neumann's, which is based on the hypothesis of repeatability of measurements.[1] A major difficulty is obtaining the transformation of states due to the measurement of observables. For discrete observables, an operational approach was introduced by Schwinger[2] and Haag and Kastler.[3]

The other approach relies on seeking analogy between Kolmogorov's measure-theoretic formulation of classical probability theory[4] and von Neumann's Hilbert-space formulation of quantum mechanics.[5] What are supposed to correspond to "observables" in classical probability theory are the random variables, and the "states" of quantum theory are

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regarded as analogs of the probability measures. Again, this approach faces fundamental difficulties. Basic physical quantities (e.g., joint probability distribution functions, conditional expectations) are realizable only under very restricted conditions—which is unnatural (e.g., conditional expectations exist if and only if the observable has a discrete spectrum[6,7]; probability distributions exist if and only if the observables commute[8]).

The classical probability theory, as formulated by Kolmogorov, begins with the sample space (Ω, m), a standard Borel space. If R_B, C_p, and f denote, respectively, the space of real, bounded Borel measures on Ω, the cone of positive measures in R_B, and the functional $\langle f, \mu\rangle = \mu(\Omega)$, then (R_B, C_p, f) comprises a state space. The set of measurable observables is defined as comprising those whose values are $B(\Omega)$, where $B(\Omega)$ is the space of bounded Borel functions on Ω. A random variable is defined to be a Borel map $\alpha: \Omega \to X$, where (X, A) is a Borel space (usually the real line), and where X is a set with a σ-field A of subsets of X, and a mapping $a: A \to R_B^*$. For all states normalized by the probability law μ on Ω, the image law is defined as the probability measure v on X given by $v(E) = \mu(\alpha^{-1}E)$ for all $E \in A$. There is a one-to-one embedding of the space $B(\Omega)$ on Ω into R_B^* defined by

$$\langle f, \mu\rangle = \int_\Omega f(\omega)\, \mu\, d\omega, \qquad \forall \mu \in R_B$$

Then,

$$v(E) = \langle a(E), \mu\rangle$$

where the right-hand side is the distribution of the observable $a(\cdots)$ in the state μ.

According to the conventional formulation of quantum theory, a state is defined as a positive operator on a complex Hilbert space H such that the trace $[\rho]$ is finite. An observable is defined as a self-adjoint operator p on H. Alternatively, using the spectral theorem, an observable is a projection-valued measure $m(\cdots)$ on the σ-field of Borel subsets of the real line R. If the system is in state ρ, then the probability that the observable takes values in v is given by

$$P(\rho, R, v) = \mathrm{tr}[\rho m(v)]/\mathrm{tr}[\rho]$$

A theory encompassing the classical and quantum probability concepts can be developed by taking as a state space the triplet (B_R, C_c, f_l), where B_R is a real Banach space, C_c is a closed cone generating B_R, and f_l is a linear functional such that $\langle f_l, b\rangle = \| b \|$, $\forall b \in B_R$. A state is defined as a nonzero element of C_c, while an observable is the triplet (X, A, a), where A

is a σ-field of subsets of X, and a is a mapping, $a: A \to B_K$ such that, for every countable class $\{E_i\}$ of pairwise disjoint sets in A:

(i) $a(X) = f_l$.

(ii) $0 \leqslant a(E) \leqslant a(X), \quad \forall E \in A$.

(iii) $a(\sum_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} a(E_i)$.

None of the conventional methods of quantum mechanics is capable of resolving fundamental problems in standard quantum mechanics (e.g., restriction to forces derivable from a potential). A more general formulation with a clear physical interpretation is therefore necessary. This is the objective of our paper. It should be emphasized, however, that the presentation below is not to be taken as suggesting that quantum mechanics is a random process.

In order to give as clear a physical meaning as possible to the mechanics, we shall take the approach in which classical ideas still have meaning. Several attempts to obtain quantum mechanics using classical notions have have been made.[9,10,13,14]

Either a Brownian model is used as the basis for the construction, or a semideductive stochastic approach is taken. The second approach[9,10] is more complete than the first. However, the crucial connection between the kinematic and dynamical quantities is missing, and thus important considerations, such as the effect of non-Markovian terms, cannot be taken into account. The lack of a general force-kinematics relationship, the inability to account for the non-Markovian terms, and the neglect of spin constitute some of the limitations in previous work. These limitations are eliminated in this paper.

2. THE NEW FORMULATION

Assuming that the total force acting on the system under investigation can be considered as a sum of deterministic and nondeterministic components, then we can also consider the velocity as being of two such components. Since irrversibility is a fundamental truth, we require that our basic equation have this property. Finally, we propose a dynamical law which reduces to Newtonian mechanics for deterministic systems. Hence our fundamental equations are

$$\mathbf{F} = m\mathbf{a} \tag{1a}$$

$$\mathbf{V}_d^* = -\mathbf{V}_d, \qquad \mathbf{V}_s^* = \mathbf{V}_s \tag{1b}$$

$$\mathbf{V} = \mathbf{V}_s + \mathbf{V}_d \tag{1c}$$

where $\mathbf{F}$, m, $\mathbf{a}$, and $\mathbf{V}$, denote, respectively, the force, mass, acceleration, and velocity; $\mathbf{V}_d$, $\mathbf{V}_s$ denote the deterministic and stochastic velocities, respectively; and $\mathbf{V}_d^*$, $\mathbf{V}_s^*$ are the corresponding quantities under time-reversal.

We now need an operator $\mathscr{D}$ corresponding to d/dt in deterministic mechanics. For a functional $f(\mathbf{x}, t)$, we assume the possibility of a Taylor expansion; thus[11] (see also Ref. 12):

$$f(x(t+\Delta t), t+\Delta t) = f(x(t), t) + \Big\{\sum_i [x_i(t+\Delta t) - x_i(t)]\,\partial_i + \tfrac{1}{2}\sum_{i,j}[x_i(t+\Delta t) - x_i(t)][x_j(t+\Delta t) - x_j(t)\,\partial_i\partial_j] + \cdots\Big\} f(\mathbf{x}(t), t) \tag{2}$$

where the set $\{x_i\}$ is the collection of the components of $\mathbf{x}$. Then,

$$\begin{aligned}(1/\Delta t)[f(\mathbf{x}(t+\Delta t), t+\Delta t) - f(\mathbf{x}(t), t)] &= \Big\{(\partial/\partial t) + (1/\Delta t)\sum_i [x_i(t+\Delta t) - x_i(t)]\,\partial_i \\ &\quad + (1/2\,\Delta t)\sum_{i,j}[x_i(t+\Delta t) - x_i(t)][x_j(t+\Delta t) - x_j(t)]\,\partial_i\partial_j \\ &\quad + \cdots\Big\} f(\mathbf{x}(t), t)\end{aligned} \tag{2a}$$

If we take the mean of Eq. (2a) and consider $\Delta t \to 0$, then we have

$$\begin{aligned}\mathscr{D}f &\equiv \lim_{\Delta t\to 0} (1/\Delta t)\langle f(\mathbf{x}(t+\Delta t), t+\Delta t) - f(\mathbf{x}(t), t)\rangle \\ &= \Big[(\partial/\partial t) + \sum_i v_i\partial_i + \sum_{i,j} D_{ij}\partial_i\partial_j + \cdots\Big] f(\mathbf{x}, t)\end{aligned} \tag{3}$$

where the angular brackets denote mean in the sense of conditional expectation, and where $2D_{ij}$ is the second-order moment of the distribution, divided by Δt. Assuming that D_{ij} is diagonal, then we have

$$\mathscr{D}f = (\partial f/\partial t) + \mathbf{V}\cdot\nabla f + D\,\nabla^2 f + \cdots \tag{4a}$$

$$\mathscr{D}^*f = -(\partial f/\partial t) + \mathbf{V}^*\cdot\nabla f + D^*\,\nabla^2 f + \cdots \tag{4b}$$

Because $\mathscr{D}^*$ becomes the negative of the total derivative for the deterministic case, it is called the mean backward derivative. Correspondingly, $\mathscr{D}$ is the mean forward derivative.

The following useful relations and definitions follow at once:

$$\mathbf{V}_d = \tfrac{1}{2}(\mathscr{D} - \mathscr{D}^*)\,\mathbf{x} \equiv \mathscr{D}_d\mathbf{x} \tag{5a}$$

$$\mathbf{V}_s = \tfrac{1}{2}(\mathscr{D} + \mathscr{D}^*)\,\mathbf{x} \equiv \mathscr{D}_s\mathbf{x} \tag{5b}$$

$$\mathscr{D}x_i = v_i \tag{6a}$$

$$\mathscr{D}^*x_i = v_i^* \tag{6b}$$

Now

$$\mathbf{F} = m(\mathbf{a}_d + \mathbf{a}_s) \tag{7}$$

We can write

$$F_d/m = \lambda_1\mathbf{a}_d - \lambda_2\mathbf{a}_s \tag{7a}$$

hence,

$$F_s/m = \mathbf{a}_d(1 - \lambda_1) + \mathbf{a}_s(1 + \lambda_2) \tag{7b}$$

The requirement that Newtonian mechanics be recoverable yields

$$\lambda_1 \equiv 1 \tag{7c}$$

So, we have

$$\mathbf{F}_d = m\mathbf{a}_d - \lambda_2\mathbf{a}_s m \tag{8a}$$

$$\mathbf{F} = \mathbf{F}_d + m\mathbf{a}_s(1 + \lambda_2) \tag{8b}$$

These are the fundamental equations of the theory.

Rewriting (8), we have

$$m[\mathscr{D}_d\mathbf{V}_d - \lambda_2\mathscr{D}_s\mathbf{V}_s] = \mathbf{F}_d$$
$$\mathscr{D}_d\mathbf{V}_s + \mathscr{D}_s\mathbf{V}_d = 0$$

which give

$$(\partial\mathbf{V}_d/\partial t) + (\mathbf{V}_d\cdot\nabla)\,\mathbf{V}_d - D_-\nabla^2\mathbf{V}_d - \lambda_2(\mathbf{V}_s\cdot\nabla)\,\mathbf{V}_s - \lambda_2 D_+\nabla^2\mathbf{V}_s = \mathbf{F}_d/m \tag{9a}$$

$$(\partial\mathbf{V}_s/\partial t) + (\mathbf{V}_d\cdot\nabla)\,\mathbf{V}_s + (\mathbf{V}_s\cdot\nabla)\,\mathbf{V}_d + D_+\nabla^2\mathbf{V}_d - D_-\nabla^2\mathbf{V}_s = 0 \tag{9b}$$

where

$$D_- \equiv \tfrac{1}{2}(D^* - D), \qquad D_+ \equiv \tfrac{1}{2}(D^* + D) \tag{9c}$$

Let $\mathscr{L}_D$ and $\mathscr{L}_S$ denote. respectively, the deterministic-part and the stochastic-part operators. Let S and R be defined by

$$\mathbf{V}_d = 2\alpha\,\nabla S \tag{10a}$$

$$\mathbf{V}_s = 2\alpha\,\nabla R \tag{10b}$$

where $S(\mathbf{x}, t)$ and $R(\mathbf{x}, t)$ are real functions of $\mathbf{x}$ and t, and where α is yet to be determined.

As shown in Section 3 below, the quantity $\psi \equiv \exp(R + iS)$ is the associated wave function. We can, from standard quantum mechanics, obtain the quantum force as

$$F = \hbar^2/2m)(\partial\psi/\partial x)^2 \tag{10c}$$

and the mean acceleration as

$$m(d^2/dt^2)\langle x\rangle = \iiint_{-\infty}^{\infty} \psi\psi^*(-\partial E/\partial x)\, d\mathbf{x} \tag{10d}$$

where E is the potential energy. Taking the form of (10c) and (10d) to hold for the more general formulation, then, on using (8b), we obtain the following expression for λ_2 after a straight forward manipulation:

$$\lambda_2 = (1/ma_s)(\alpha^2/2m)(\partial\psi_s/\partial x)^2 - 1 \tag{11a}$$

where

$$ma_s = (\alpha^2/2m)[\mathscr{L}_S(\partial/\partial x)\exp(R + iS)]^2$$

ψ_s is given by

$$(\alpha^2/2m)(\partial\psi_s/\partial x)^2 = -\iiint_{-\infty}^{\infty} \psi_s\psi_s^*(\alpha^2/2m)(\partial\psi_s/\partial x)^2\, d\mathbf{x}$$

and where we have used α for the constant corresponding to h in the general case. So,

$$(\partial\mathbf{V}_d/\partial t) + (\mathbf{V}_d\cdot\nabla)\,\mathbf{V}_d - D_-\nabla^2\mathbf{V}_d - \lambda_2(\alpha, R, S)(\mathbf{V}_s\cdot\nabla)\,\mathbf{V}_s - \lambda_2(\alpha, R, S)\,D_+\nabla^2\mathbf{V}_s = F_d/m \tag{11b}$$

and

$$(\partial\mathbf{V}_s/\partial t) + (\mathbf{V}_d\cdot\nabla)\,\mathbf{V}_s + (\mathbf{V}_s\cdot\nabla)\,\mathbf{V}_d + D_+\nabla^2\mathbf{V}_d - D_-\nabla^2\mathbf{V}_s = 0 \tag{11c}$$

Equations (11) are the basic equations of our theory.

3. IMPORTANT SPECIAL CASE: THE SCHRÖDINGER EQUATION

We consider the special case where: (i) the coefficients D_+ and D_- depend only on time, (ii) the deterministic force is derivable from a potential, (iii) the velocity $\mathbf{V}$ is irrotational. Then, we have

$$(\partial\mathbf{V}_d/\partial t) + \nabla[\tfrac{1}{2}\mathbf{V}_d^2 - D_-\nabla\cdot\mathbf{V}_d - \tfrac{1}{2}\lambda_2\mathbf{V}_s^2 - \lambda_2 D_+\nabla\cdot\mathbf{V}_s = -\nabla\phi \tag{12a}$$

$$(\partial\mathbf{V}_s/\partial t) + \nabla[\mathbf{V}_d\cdot\mathbf{V}_s + D_+\nabla\cdot\mathbf{V}_d - D_-\nabla\cdot\mathbf{V}_s] = 0 \tag{12b}$$

where $-\nabla\phi = F_d/m$. Now, use $\mathbf{V}_d = 2\alpha\nabla S$ and $\mathbf{V}_s = 2\alpha\nabla R$, and put

$$\psi \equiv \exp(R + iS) \tag{13}$$

Then, we have

$$\begin{aligned} i\alpha\,\partial\varphi/\partial t = \tfrac{1}{2}\psi\{\phi + \nabla^2(\ln\psi)(-2\alpha^2)[(D_+/\alpha) - (iD_-/\alpha) - 1] \\ \times (1 - \lambda_2)(D_+\nabla\cdot\mathbf{V}_s + \tfrac{1}{2}\mathbf{V}_s^2)] - \alpha^2\nabla^2\psi \end{aligned} \tag{14}$$

Choose $\lambda_2 \equiv 1$, $D_+ = \alpha =$ const, and $D_- = 0$. This choice of quantities simplifies the nonlinear equation (14), reducing it to a linear equation, thereby decoupling the Schrödinger equation from its complex conjugate. Thus we have

$$i\alpha\,\partial\psi/\partial t = [\tfrac{1}{2}\phi - \alpha^2\nabla^2]\,\psi \tag{15}$$

But this is the Schrödinger equation. Thus, we identify α as

$$\alpha \equiv \hbar/2m \tag{16}$$

4. FURTHER SPECIAL CASES: BROWNIAN MOTION AND CONSERVATION EQUATIONS

In Brownian motion, the basic quantity is the probability density P:

$$P = \psi\psi^* = e^{2R} \tag{17}$$

Using the same assumptions as those needed to obtain Schrödinger's equation gives

$$\mathbf{V}_s = \nabla(\alpha\ln P) = \alpha(\nabla p)/p \tag{18}$$

which is Einstein's equation of Brownian motion. Combining (18) with (12) yields

$$(\partial p/\partial t) + (\nabla\cdot\mathbf{V}_d)\,p = 0 \tag{19}$$

which is the equation of continuity, and

$$\partial p/\partial t = -\nabla\cdot\mathbf{V}P + D\,\nabla^2 P \tag{20}$$

which is the Fokker–Planck equation, and

$$D \equiv D_+ - D_- \tag{20a}$$

For completeness, we add the equation of conservation of energy,

$$\tfrac{1}{2}m\mathbf{V}_d^2 + \tfrac{1}{2}m\mathbf{V}_s^2 + \Phi = -\hbar\,\partial S/\partial t \tag{21}$$

where $\Phi = m\phi$.

A phenomenological description of Brownian motion is through the Langevin equation

$$m\, d\mathbf{V}/dt = -m\beta\mathbf{V} + \mathbf{F}_d + \mathbf{F}_s(t) \tag{22}$$

For $\beta t \gg 1$, the acceleration is small, and so $d\mathbf{V}/dt \approx 0$. But also $\langle F_s(t)\rangle = 0$. Hence,

$$\mathbf{V} \approx \mathbf{F}_d/m\beta \tag{23}$$

Using (23) in (20) yields

$$\partial p/\partial t = -\nabla \cdot (\mathbf{F}_d/m\beta)\, p + D\, \nabla^2 p \tag{24}$$

which is the Smoluchowski equation.

Uncertainty relations. Let

$$\Delta x_i = x_i - \langle x_i\rangle, \qquad \Delta v_i = v_i - \langle v_i\rangle \tag{25}$$

Then,

$$\langle(\Delta x_i)^2\rangle\langle(\Delta v_i)^2\rangle \geqslant D^2 \tag{26}$$

In the quantum mechanical case, $D = \alpha$, and

$$\langle(\Delta x_i)^2\rangle\langle(\Delta v_i)^2\rangle \geqslant \alpha^2 \tag{27}$$

5. CONCLUSION

We have demonstrated that a stochastic theory approach could be used as the foundation for quantum mechanics. The approach has the advantages of physical clarity and a minimum of postulates, in addition to not being being limited to forces derivable from a potential. When the new formulation is used for extended rigid particles, then a generalized Schrödinger equation for integral or half-integral spins is obtained. This formulation opens new possibilities, particularly in the investigation of many problems not within the scope of standard quantum mechanics.

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